PMM U.S.S.R.,Vol.49,No.3,pp. 380-386,1985
Printed in Great Britain
0021-8928/85 $10.00+0.00
Pergamon Journals Ltd.

SINGULAR PROBLEMS OF THE THEORY OF ELASTICITY FOR CRACKS PERPENDICULAR TO THE BOUNDARY SEPARATING TWO MEDIA*

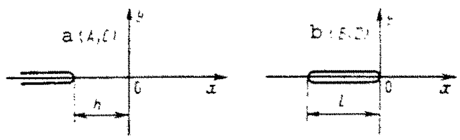
N.D. ZHEKOV and V.D. KULIEV

The Wiener-Hopf method is used to construct exact solutions for the problems of cracks perpendicular to the plane boundary separating two different, homogeneous isotropic elastic media. The solutions are constructed for the follwoing cases: semi-infinite cracks with the tip situated at a finite distance from the interface (a normal fracture crack is covered in problem A and a longitudinal shear crack in problem C): a crack of finite length with one of its ends lying at the interface (a normal fracture crack is covered in problem B and a longitudinal shear crack in problem D).

Problem B was solved earlier in /1/. A different method of factorization /2—5/ used below leads to a relatively simple construction of the solution for problem B, and also for problems A, C, D.

1. Formulation of the problem. Let two isotropic, homogeneous elastic half-spaces with different elastic properties be rigidly bonded to each other along the plane $x = 0$. A crack of length l is situated along the negative part of the x axis at a distance h from the interface. This problem was solved in /6/ for a body of finite linear dimensions using the method of finite elements. Below we describe two limiting cases of this problem: a) $l \to \infty$ (problems A, C for normal fracture cracks and longitudinal shear cracks respectively, figure a);

b) $h \to 0$ (problems B, D also for normal fracture cracks and longitudinal shear cracks respectively, figure b).



We assume that the values of the elastic constants E_1, ν_1 for the first material are specified in the left-hand half-plane (for $x < 0$) and in the right-hand half-plane (for $x > 0$) E_2, ν_2 are given for the second material.

The boundary conditions for problems A and B are

$$\theta = \pm\pi/2, \ [u_\theta] = [u_r] = 0, \ [\sigma_\theta] = [\tau_{r\theta}] = 0 \tag{1.1}$$
$$\theta = 0, \ \tau_{r\theta} = 0, \ u_\theta^{\pm} = 0$$
$$\theta = \pm\pi, \ \tau_{r\theta} = 0$$

and we also have

$$\theta = \pm\pi, \ 0 < r < h, \ u_\theta = 0 \tag{1.2}$$
$$r > h, \ \sigma_\theta = 0 \ \text{for problem} \ A$$
$$\theta = \pm\pi, \ 0 < r < l, \ \sigma_y(x) = -\sigma(x) \tag{1.3}$$
$$r > l, \ u_\theta = 0 \ \text{for problem} \ B$$

The boundary conditions for problems C and D are

$$\theta = \pm\pi/2, \ [\sigma_{\theta 3}] = 0, \ [w] = 0 \tag{1.4}$$
$$\theta = 0, \ w = 0$$

and we also have

$$\theta = \pm\pi, \ r > h, \ \sigma_{\theta 3} = 0 \tag{1.5}$$
$$0 < r < h, \ w = 0 \ \text{for problem} \ C$$
$$\theta = \pm\pi, \ 0 < r < l, \ \sigma_{\theta 3} = \tau(x) \tag{1.6}$$
$$r > l, \ w = 0 \ \text{for problem} \ D$$

---

*Prikl.Matem.Mekhan.,49,3,492-500,1985

In addition to the conditions (1.1), (1.2), (1.4), (1.5) we specify a condition as $r\to\infty$ for problems A and C. It is clear that when $h\to 0$, we arrive at the Zak-Williams problem /7/. When $h\neq 0$, the exact Zak-Williams solution must be realized in the form of a given asymptotic form as $r\to\infty$.

For problem A we have /2, 7/

$$\sigma_\theta = K^\circ[(2+\lambda)\cos\lambda\theta + B\cos(2+\lambda)\theta] \tag{1.7}$$
$$\sigma_r = K^\circ[(2-\lambda)\cos\lambda\theta - B\cos(2+\lambda)\theta]$$
$$\tau_{r\theta} = K^\circ[\lambda\sin\lambda\theta + B\sin(2+\lambda)\theta],\quad -\pi/2<\theta<\pi/2$$
$$\frac{\partial u_\theta}{\partial r} = -K^\circ\frac{(1-\nu_1)}{2\mu_1}\sin\lambda\pi\,\frac{B(3+2\lambda)-(2\lambda^2+5\lambda+2)}{\lambda(2+\lambda)+\sin^2\lambda\pi/2},\quad \pi/2<\theta<\pi$$
$$B=\frac{1}{1+k_1}[(2+3\lambda)k_1-(1+2\lambda)k_2+1+\lambda]$$
$$k_1=\frac{k-1}{4(1-\nu_1)},\quad k_2=\frac{1-\nu_2}{1-\nu_1}k,\quad k=\frac{\mu_1}{\mu_2},\quad K^\circ=K_1\frac{(1+\lambda)r^\lambda}{\sqrt{2\pi}}$$

and for problem C we have

$$\sigma_{\theta 3}=K_{\rm III}{}^\circ\cos(\delta+1)\theta,\quad \sigma_{r3}=K_{\rm III}{}^\circ\sin(\delta+1)\theta,\quad |\theta|<\pi/2 \tag{1.8}$$
$$w=K_{\rm III}\frac{r^{\delta+1}}{\mu_1\sqrt{2\pi}(\delta+1)},\quad \theta=\pi$$
$$K_{\rm III}^\circ=K_{\rm III}\frac{r^\delta}{\sqrt{2\pi k}}$$

Here $K_{\rm I}$, $K_{\rm III}$ are the stress intensity coefficients assumed given for the above problems, $\sigma_\theta, \sigma_r, \tau_{r\theta}, \sigma_{\theta 3}, \sigma_{r3}$ are the stresses, ν and μ are Poisson's ratio and the shear modulus respectively, and λ is the unique, real root of the equation

$$\cos\pi\lambda = a + b(\lambda+1)^2 \tag{1.9}$$
$$\left(a=\frac{2k_1^2-2k_1k_2-2k_1-k_2+1}{2(k_2-k_1)(k_1+1)},\quad b=\frac{2k_1}{k_1+1}\right)$$

lying in the interval $(-1, 0)$. The degree of singularity of the stresses δ is determined by the formula

$$\delta=-2\pi^{-1}\operatorname{arctg}\sqrt{k}\ (-1<\delta<0) \tag{1.10}$$

In the case of problem B and D, the stresses tend to zero as $r\to\infty$.

The solutions of the problems of the theory of elasticity sought here must satisfy the boundary conditions (1.1)–(1.6), and conditions at infinity. The following asymptotic form must be realized near the ends of the cracks ($\varepsilon\ll 1$):

$$\sigma_\theta(r,\pi)=\frac{k_{\rm I}}{\sqrt{2\pi\varepsilon}},\quad \varepsilon=1-r \tag{1.11}$$
$$\frac{\partial u_\theta(r,\pi)}{\partial r}=-\frac{2(1-\nu_1^2)k_{\rm I}}{E_1\sqrt{2\pi\varepsilon}},\quad \varepsilon=r-1$$

for problem A ($K_{\rm I}$ is the stress intensity factor to be determined), and

$$\sigma_{\theta 3}=-\frac{k_{\rm III}}{\sqrt{2\pi\varepsilon}},\quad \varepsilon=1-r \tag{1.12}$$
$$\frac{\partial w(r,\pi)}{\partial r}=\frac{k_{\rm III}}{\mu_1\sqrt{2\pi\varepsilon}},\quad \varepsilon=r-1$$

for the problem C.

In the case of problems B and D, the corresponding asymptotic curve must be realized for cracks perpendicular to the interface, near the tip of the crack lying on the interface: relations (1.7) are taken as the asymptotic expression for problem B at the crack tip, and (1.8) for D; for the other tip of the crack not lying on the boundary, we use relation (1.11) for B and (1.12) for D.

2. Derivation of the Wiener-Hopf equations.

Problem A and B. Applying the integral Mellin transformation /8/ to the equations of equilibrium and compactness of the plane problem of the theory of elasticity, we arrive at the fourth-order ordinary differential equation /8/ whose solution will be sought in the form

$$\sigma_\theta^*(p,\theta)=A_1\cos(p+1)\theta+A_2\cos(p-1)\theta+ \tag{2.1}$$
$$A_3\sin(p+1)\theta+A_4\sin(p-1)\theta,\quad 0\leqslant\theta\leqslant\pi/2$$
$$\sigma_\theta^*(p,\theta)=B_1\cos(p+1)\theta+B_2\cos(p-1)\theta+$$
$$B_3\sin(p+1)\theta+B_4\sin(p-1)\theta,\quad \pi/2\leqslant\theta\leqslant\pi$$

(A_i, B_i are unknown functions of the parameter p). The functions σ_r^* and $\tau_{r\theta}^*$ are given in terms of σ_θ^* as follows:

$$\tau_{r\theta}^*=\frac{1}{p-1}\frac{d\sigma_\theta^*}{d\theta},\quad p\sigma_r^*=\frac{1}{p-1}\frac{d^2\sigma_\theta^*}{d\theta^2}-\sigma_\theta^* \tag{2.2}$$

Applying the Mellin transform to Hooke's law we obtain

$$\left(\frac{\partial u_r}{\partial r}\right)^* = \frac{1+\nu_j}{E_j}\left[(1-\nu_j)\sigma_r^* - \nu_j\sigma_\theta^*\right] \tag{2.3}$$

$$\left(\frac{\partial u_\theta}{\partial r}\right)^* = \frac{1+\nu_j}{E_j(p+1)}\left[2p\tau_{r\theta}^* + (1-\nu_j)\frac{d\sigma_r^*}{d\theta} - \nu_j\frac{d\sigma_\theta^*}{d\theta}\right], \quad j=1,2$$

Using (2.1)–(2.3), (1.1) we obtain a system of equations in A_i, B_i. We shall write the solution of this system in the form

$$\begin{aligned}
&A_3 = A_4 = 0 \\
&A_1 = B_3 (k_1+1)[k_1(k_1-k_2)(2p+1)\sin p\pi]^{-1} \\
&B_1 = B_3 [2k_1(p\cos p\pi - \sin^2 p\pi/2) - 1][k_1(2p+1)\sin p\pi]^{-1} \\
&A_2 = B_3 [(k_1-k_2)(2p+1) + p(k_1+1)][\Delta(p)\sin p\pi]^{-1} \\
&B_2 = B_3 [2k_1(k_1-k_2)(p+1)(p\cos p\pi + \sin^2 p\pi/2) - \\
&\quad (k_1+1)\cos p\pi + (k_1-k_2)(p+2\sin^2 p\pi/2)][\Delta(p)\sin p\pi]^{-1} \\
&B_4 = B_3 \{(k_1-k_2)[k_1 p(2p+1) - (k_1+1)] - (k_1+1)\}[\Delta(p)]^{-1} \\
&\Delta(p) = k_1(k_1-k_2)(2p+1)(p-1)
\end{aligned} \tag{2.4}$$

In accordance with (1.2) and (2.1)–(2.4), we now arrive at the homogeneous functional Wiener-Hopf equation for problem A

$$(p+\lambda+1)\Phi_A^+(p) = {}^1/_2 G_A(p)K_A(p)\Phi_A^-(p) \tag{2.5}$$

$$\Phi_A^+(p) = \frac{E_1}{4(1-\nu_1^2)}\int_1^\infty \left(\frac{\partial u_\theta}{\partial s}\right)_{\theta=\pi} s^p\, ds \tag{2.6}$$

$$\begin{aligned}
&\Phi_A^-(p) = \int_0^1 (\sigma_\theta)_{\theta=\pi} s^p\, ds \\
&G_A(p) = \operatorname{ctg} p\frac{\pi}{2}\sin^2 p\frac{\pi}{2}\operatorname{tg}(p+\lambda+1)\pi\,[\gamma(p)]^{-1} \\
&K_A(p) = (p+\lambda+1)\operatorname{ctg}(p+\lambda+1)\pi \\
&\gamma(p) = \sin^2 p\frac{\pi}{2} - p^2\frac{k_1}{k_1+1} - \frac{k_2+1}{4(k_1+1)(k_2-k_1)}
\end{aligned} \tag{2.7}$$

The function $\Phi_A^-(p)$ is analytic in the right-hand half-plane $\operatorname{Re} p > -1$, and the function $\Phi_A^+(p)$ is analytic in the left-hand half-plane $\operatorname{Re} p < -(\lambda+1)$.

Using (1.3) we obtain, in the same manner as (2.5), the Wiener-Hopf equation for problem B

$$\left(\frac{p}{\lambda-1} - 1\right)\Phi_B^-(p) = G_B(p)K_B(p)[F_B(p) + \Phi_B^+(p)] \tag{2.8}$$

where

$$\Phi_B^-(p) = \frac{E_1}{4(1-\nu_1^2)}\int_0^1 \left(\frac{\partial u_\theta}{\partial r}\right)_{\theta=\pi} s^p\, ds, \quad \Phi_B^+(p) = \int_1^\infty (\sigma_\theta)_{\theta=\pi} s^p\, ds \tag{2.9}$$

$$G_B(p) = \operatorname{ctg} p\frac{\pi}{2}\sin^2 p\frac{\pi}{2}\operatorname{tg}\left(\frac{p}{\lambda-1}+1\right)\frac{\pi}{2}[\gamma(p)]^{-1}$$

$$K_B(p) = \frac{1}{2}\left(\frac{1}{\lambda-1}+1\right)\operatorname{ctg}\left(\frac{p}{\lambda-1}+1\right)\frac{\pi}{2}, \quad F_B(p) = \int_0^1 \sigma_y(s)s^p\, ds$$

The function $\Phi_B^-(p)$ is analytic in the right half-plane $\operatorname{Re} p > -(\lambda+1)$, $\Phi_B^+(p)$ when $\operatorname{Re} p < 0$.

Problems C and D. The Wiener-Hopf equations for problems C and D are obtained exactly as before. Applying the integral Mellin transform to the relations of the theory of elasticity for complex shear /5/, we obtain a second-order differential equation and seek its solution in the form

$$\begin{aligned}
&W = A_1\cos p\theta + A_2\sin p\theta, \quad 0 \leqslant \theta \leqslant \pi/2 \\
&W = B_1\cos p\theta + B_2\sin p\theta, \quad \pi/2 \leqslant \theta \leqslant \pi \\
&W \equiv \left(\frac{\partial w}{\partial r}\right)^* = \int_0^\infty \frac{\partial w}{\partial r} r^p\, dr
\end{aligned} \tag{2.10}$$

Here A_i, B_i are functions of p to be determined. The functions $\sigma_{r3}^*(p,\theta)$, $\sigma_{\theta 3}^*(p,\theta)$ are written in terms of $W(p,\theta)$ in the form

$$\sigma_{r3}^* = \mu_j W, \quad \sigma_{\theta 3}^* = -\frac{\mu_j}{p}\frac{dW}{d\theta} \tag{2.11}$$

Conditions (1.4) yield a system of equations whose solution will be

$$A_1=0,\ A_2=B_1\frac{2k}{(k-1)\sin p\pi} \qquad B_2 = 2B_1\,(k\sin^2 p\pi/2+\cos^2 p\pi/2)[(k-1)\sin p\pi]^{-1} \tag{2.12}$$

Further, using the conditions (1.5) we obtain the Wiener-Hopf equation for problem C

$$K_C(p)\Phi_C^-(p) = (p+\delta+1)G_C(p)\Phi_C^+(p) \tag{2.13}$$

where

$$\Phi_C^-(p)=\int_0^1(\sigma_{\theta 3})_{\theta=\pi}s^p\,ds,\quad \Phi_C^+(p)=\mu_1\int_1^\infty\left(\frac{\partial w}{\partial s}\right)_{\theta=\pi}s^p\,ds \tag{2.14}$$

$$G_C(p)=\frac{(k\,\mathrm{tg}^2 p\pi/2-1)\,\mathrm{ctg}\,(p+\delta+1)}{(k+1)\,\mathrm{tg}\,p\pi/2}$$

$$K_C(p)=(p+\delta+1)\,\mathrm{ctg}\,(p+\delta+1)\pi$$

The function $\Phi_C^+(p)$ is analytic in the left-hand half-plane $\mathrm{Re}\,p<-\delta-1$, and $\Phi_C^-(p)$ is analytic for $\mathrm{Re}\,p>-1$.

Problem D. Introducing the functions

$$\Phi_D^-(p)=\mu_1\int_0^1\left(\frac{\partial w}{\partial s}\right)_{\theta=\pi}s^p\,ds,\quad \Phi_D^+(p)=\int_1^\infty(\sigma_{\theta 3})_{\theta=\pi}s^p\,ds \tag{2.15}$$

we obtain from (1.6)

$$K_D(p)\,[\Phi_D^+(p)+F_D(p)]=\frac{1}{2}\left(\frac{p}{\delta-1}+1\right)G_D(p)\,\Phi_D^-(p) \tag{2.16}$$

where

$$G_D=\left(k\,\mathrm{tg}^2 p\frac{\pi}{2}-1\right)\mathrm{ctg}\left(\frac{p}{\delta-1}-1\right)\frac{\pi}{2}\left[(k-1)\,\mathrm{tg}\,p\frac{\pi}{2}\right]^{-1} \tag{2.17}$$

$$K_D(p)=\frac{1}{2}\left(\frac{p}{\delta-1}-1\right)\mathrm{ctg}\left(\frac{p}{\delta-1}+1\right)\frac{\pi}{2},\quad F_D(p)=\int_0^1\tau(s)\,s^p\,ds$$

The functions $\Phi_D^+(p)$ and $\Phi_D^-(p)$ are analytic for $\mathrm{Re}\,p<0$ and $\mathrm{Re}\,p>-\delta-1$ respectively.

3. Solution of the boundary value problems.

Problem A. The Wiener-Hopf equation (2.5) is valid within the strip $-1<\mathrm{Re}\,p<-(\lambda+1)$, $-\infty<\mathrm{Im}\,p<\infty$. The function $G_A(p)$ has the following properties: it is regular and has no zeros within the strip $-1<\mathrm{Re}\,(p)\leqslant-(\lambda+1)$, provided that $k<1$. When $k>1$, the function $G_A(p)$ is regular and has no zeros within the strip $-(\lambda+3/2)<\mathrm{Re}\,(p)\leqslant-(\lambda+1)$. When $\mathrm{Im}\,(p)\to\pm\infty$ $\mathrm{Re}\,(p)=-\lambda-1$, $G_A(p)\to 1$. The function $\gamma(p)$ has a first-order zero at the point $p=-(\lambda+1)$, and the zero of this function is a root of the characteristic equation (1.9).

Let us denote the regions situated to the left and right of the contour L_A (L_A : $\mathrm{Re}\,p=-(\lambda+1)$, $-\infty<\mathrm{Im}\,p<\infty$) by D^+ and D^-, respectively. The function $G_A(p)$ can be written in the form /9, 10/

$$G_A(p) = G_A^+(p)\,G_A^-(p) \tag{3.1}$$

$$\exp\left[\frac{1}{2\pi i}\int_{L_A}\frac{\ln G_A(p)}{t-p}\,dt\right]=\begin{cases}G_A^+(p), & p\in D^+\\ G_A^-(p), & p\in D^-\end{cases} \tag{3.2}$$

The functions G_A^+ and G_A^- in (3.1) are analytic, have no zeros in the regions D^+ and D^-, respectively, and tend to unity as $p\to\infty$. We use the following representation in factorizing the function $K_A(p)$:

$$K_A(p) = K_A^+(p)K_A^-(p) \tag{3.3}$$

$$K_A^\pm(p)=\frac{\Gamma[1\mp(p+\lambda+1)]}{\Gamma[1/2\mp(p+\lambda+1)]}$$

where $\Gamma(p)$ is the gamma function.

The function $K_A^+(p)$ is regular and has no zeros when $\mathrm{Re}\,p<-(\lambda+1/2)$, while the function $K_A^-(p)$ is regular and has no zeros when $\mathrm{Re}\,p>-(\lambda+3/2)$. Moreover,

$$K_A^\pm(p)\sim\sqrt{\mp p}+o(1)\quad\text{as}\quad p\to\infty \tag{3.4}$$

Using the representations (3.2), (3.3) for (2.5) we obtain

$$\frac{(p+\lambda+1)\,\Phi_A^+(p)}{K_A^+(p)\,G_A^+(p)}=\frac{1}{2}\frac{K_A^-(p)}{G_A^-(p)}\,\Phi_A^-(p) \tag{3.5}$$

The left side of this equation is analytic in D^+ and the right side is analytic in D^-. According to the principle of analytic continuation, these sides are equal to unity and the same function analytic in the whole plane. To find this function we determine the behaviour of the left and right sides of (3.5) as $p \to \infty$. Let us inspect the behaviour of the unknown $\Phi_A^+(p)$ and $\Phi_A^-(p)$ as $p \to \infty$. According to the Abel-type theorem /10/ we obtain, as $p \to \infty$, with the help of (1.11),

$$\Phi_A^-(p) \sim \frac{k_1}{\sqrt{2p}}, \quad \Phi_A^+(p) \sim -\frac{k_1}{2\sqrt{-2p}} \tag{3.6}$$

Taking into account (3.2), (3.5) we obtain

$$\Phi_A^-(p) = \frac{k_1 G_A^-(p)}{\sqrt{2}\, K_A^-(p)}, \quad \Phi_A^+(p) = \frac{k_1 K_A^+(p)\, G_A^+(p)}{2\sqrt{2}\,(p + \lambda + 1)} \tag{3.7}$$

The solution (3.7) contains the parameter k_1 which must be obtained from the condition (1.7).

Using (1.7) we have, by virtue of the Abel-type theorem $(p \to -\lambda - 1)$

$$\left(\frac{\partial u_\theta}{\partial r}\right)^* = K_1 \frac{(1-\nu_1^2)}{E_1\sqrt{2\pi}} \frac{(\lambda-1)\sin\lambda\pi\,[B(3+2\lambda) - (2\lambda^2+5\lambda+2)]}{[\lambda(2-\lambda) + \sin^2\lambda\pi/2]\,(p+\lambda+1)} \tag{3.8}$$

On the other hand, from (3.7) we obtain

$$\left(\frac{\partial u_\theta}{\partial r}\right)^* = k_1 \frac{2(1-\nu_1^2)}{E_1\sqrt{2\pi}} \frac{G_A^+(-\lambda-1)}{(p+\lambda-1)} \tag{3.9}$$

Equating (3.8) and (3.9) and changing to dimensional coordinates, we obtain

$$K_{1(A)} = K_1 \frac{(\lambda-1)\sin\lambda\pi\,[B(3-2\lambda) - (2\lambda^2-5\lambda-2)]\,h^{\lambda+1/2}}{2\,[\lambda(2-\lambda) + \sin^2\lambda\pi/2]\,G_A^+(-\lambda-1)} \tag{3.10}$$

Problem B. The Wiener-Hopf equation of problem *B* (2.8) is defined in the strip $-\lambda - 1 < \mathrm{Re}\, p < 0$. The function $G_B(p)$ (2.9) has the following properties. It is regular and has no zeros within the strip of definition of the equations. Moreover, as was shown before, when $\mathrm{Im}\, p \to \pm\infty$ $G_B(p) \to 1$, then $\gamma(p)$ has a first-order zero at the point $p = -(\lambda+1)$. We shall denote the regions situated to the left and right of the contour L_B $(L_B: -(\lambda+1) \leqslant \mathrm{Re}\, p \leqslant 0,$ $-\infty < \mathrm{Im}\, p < \infty)$, by D^- and D^+. Then G_B can be written in the form /9, 10/

$$G_B(p) = G_B^+(p)/G_B^-(p) \tag{3.11}$$

Here $G_B^-(p)$ and $G_B^-(p)$ are obtained in the same manner as $G_A^{\pm}(p)$, but using $G_B(p)$ and taking into account L_B. The function $K_B(p)$ can be written, like (3.3), in the form

$$K_B(p) = K_B^-(p) K_B^-(p) \tag{3.12}$$
$$K_B^{\pm}(p) = \Gamma\left[1 \mp \frac{1}{2}\left(\frac{p}{\lambda-1} - 1\right)\right]\left(\Gamma\left[\frac{1}{2} \mp \frac{1}{2}\left(\frac{p}{\lambda-1} + 1\right)\right]\right)^{-1}$$

Using (3.11), (3.12) we obtain from (2.8)

$$\left(\frac{p}{\lambda-1} - 1\right)\frac{\Phi_B^- G_B^-(p)}{K_B^-(p)} = F_B(p) K_B^-(p) G_B^+(p) - K_B^+(p) G_B^-(p) \Phi_B^+(p) \tag{3.13}$$

Let the function

$$\Psi_B(p) = F_B(p) K_B^+(p) G_B^-(p)\, p \tag{3.14}$$

be such that

$$\Psi_B(p) = \Psi_B^+(p) - \Psi_B^-(p) \tag{3.15}$$

$$\frac{1}{2\pi i}\int_{L_B} \frac{F_B(t) K_B^-(t) G_B^-(t)}{t\,(t-p)}\, dt = \begin{cases} \Psi_B^+(p), & p \in D^+ \\ \Psi_B^-(p), & p \in D^- \end{cases} \tag{3.16}$$

$$\frac{p+\lambda-1}{p(\lambda-1)} \frac{\Phi_B^-(p)\, G_B^-(p)}{K_B^-(p)} + \Psi_B^-(p) = \tag{3.17}$$
$$\Psi_B^+(p) - \frac{K_B^-(p)\, G_B^+(p)\, \Phi_B^+(p)}{p} \quad (p \in L_B)$$

From (1.1) it follows that $\Phi_B^-(p)$ has a first-order zero at the point $p = 0$, therefore the left and right side of (3.17) represent the analytic functions D^- and D^+ respectively. Using Liouville's theorem we obtain

$$\Phi_B^-(p) = -\frac{p(\lambda+1)\,\Psi_B^-(p)\, K_B^-(p)}{(p-\lambda+1)\, G_B^-(p)}, \quad \Phi_B^+(p) = -\frac{p\Psi_B^+(p)}{K_B^+(p)\, G_B^+(p)} \tag{3.18}$$

To find the stress intensity coefficient at the right end of the crack we shall use the asymptotic relation (3.9) obtained earlier (with the sign changed). Separating from

$\partial u_\theta/\partial r\ \Phi_B^-(p)$ according to (2.9) and equation with (3.18) we obtain, after changing to dimensional variables,

$$K_{\mathrm{I}(B)} = -\frac{4\sqrt{2}\,[\lambda(2+\lambda)+\sin^2\lambda\pi/2]}{l^{\lambda}\sin\lambda\pi\,[B(3+2\lambda)-(2\lambda^2+5\lambda+2)]}\Psi_B^-(-\lambda-1) \tag{3.19}$$

where

$$\Psi_B^-(-\lambda-1) = \frac{1}{2\pi i}\int_{L_B^*}\frac{\Psi_B(t)}{t+\lambda+1}\,dt \tag{3.20}$$

(L_B^* : $\mathrm{Re}\,p = -\lambda-1,\ -\infty < \mathrm{Im}\,p < \infty$; the point $p=-\lambda-1$ is passed on the left side along a semicircle of small radius with centre at this point).

Let us determine the stress intensity coefficient at the left end of the crack. From the Abel-type theorem /10/ we obtain, using (1.11),

$$\Phi_B^-(p) \sim \frac{k_{\mathrm{I}}}{2\sqrt{2p}} \quad (p\to\infty) \tag{3.21}$$

On the other hand we have from (3.18), as $p\to\infty$

$$\Phi_B^-(p) = \frac{\sqrt{\lambda+1}}{\sqrt{2p}}\,g_B(\sigma) \quad (p\to\infty) \tag{3.22}$$

where

$$g_B(\sigma) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}K_B^+(t)\,G_B^+(t)\,F_B(t)\,dt \tag{3.23}$$

Equating (3.22) and (3.21) we obtain, after changing to dimensional coordinates,

$$k_{\mathrm{I}(B)} = -2\sqrt{l(1+\lambda)}\,g_B(\sigma) \tag{3.24}$$

Problems C and D. In the case of problem *C* the Wiener-Hopf equation (2.13) is factorized just as in problem *A*. As a result we obtain

$$\Phi_C^-(p) = -\frac{k_{\mathrm{III}(C)}}{\sqrt{2}\,K_C^-(p)\,G_C^-(p)} \tag{3.25}$$

$$\Phi_C^+(p) = -\frac{k_{\mathrm{III}(C)}K_C^+(p)}{\sqrt{2}(p+\delta-1)\,G_C^+(p)},\quad K_C^{\pm}(p) = \frac{\Gamma[1\mp(p-\delta+1)]}{\Gamma[{}^1/_2\mp(p+\delta+1)]}$$

$$\exp\left[\frac{1}{2\pi i}\int_{L_C}\frac{G_C(t)}{t-p}\,dt\right] = \begin{cases} G_C^+(p), & p\in D^+\\ G_C^-(p), & p\in D^-\end{cases}$$

Using the Abel-type theorem /10/ we obtain from (1.8), just as in problem *A*, a relation connecting K_{III} and $k_{\mathrm{III}(C)}$

$$k_{\mathrm{III}(C)} = K_{\mathrm{III}}G_C^+(-\delta-1)\,h^{\delta+1/2} \tag{3.26}$$

After factorizing the Wiener-Hopf equation (2.16) of problem *D* carried out as in the case of Eq.(2.8) of problem *B*, we obtain

$$\Phi_D^+(p) = -\frac{p\Psi_D^+(p)\,G_D^+(p)}{K_D^+(p)}, \tag{3.27}$$

$$\Phi_D^-(p) = -\frac{2p(\delta+1)\,K_D^-(p)\,G_D^-(p)\,\Psi_D^-(p)}{(p+\delta-1)}$$

$$\exp\left[\frac{1}{2\pi i}\int_{L_D}\frac{\ln G_D(t)}{t-p}\,dt\right] = \begin{cases} G_D^+(p), & p\in D^+\\ G_D^-(p), & p\in D^-\end{cases} \tag{3.28}$$

$$(L_D:\ -\delta-1\leqslant \mathrm{Re}\,p<0,\ -\infty\leqslant \mathrm{Im}\,p<\infty)$$

$$K_D^{\pm}(p) = \Gamma\left[1\mp\frac{1}{2}\left(\frac{p}{\delta+1}-1\right)\right]\left(\Gamma\left[\frac{1}{2}\mp\frac{1}{2}\left(\frac{p}{\delta+1}-1\right)\right]\right)^{-1}$$

$$\frac{1}{2\pi i}\int_{L_D}\frac{F_D(t)\,K_D^+(t)}{t(t-p)\,G_D^+(t)}\,dt = \begin{cases} \Psi_D^+(p), & p\in D^+\\ \Psi_D^-(p), & p\in D^-\end{cases}$$

Further, using (1.8) we obtain for the right tip

$$K_{\mathrm{III}(D)} = 2\sqrt{2}\,(\delta+1)^2\,l^{-\delta}\,\Psi_D^-(-\delta-1)\,G_D^-(-\delta-1) \tag{3.29}$$

$$\Psi_D^-(-\delta-1) = \frac{1}{2\pi i}\int_{L_D^*}\frac{F_D(t)\,K_D^+(t)}{t(t+\delta+1)\,G_D^+(t)}\,dt$$

(the contour L_D^* coincides with L_B^*, after formally replacing λ by δ).

Using (1.12), we obtain the following expression in dimensional variables for the left tip of the crack:

$$k_{\text{III}(D)} = -2\sqrt{(\delta+1)l}\, g_D(\tau), \quad g_D(\tau) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \frac{F_D(t)K_D^+(t)}{tG_D^+(t)}\,dt \qquad (3.30)$$

4. Analysis of the solutions obtained.

Let us consider special cases of the general solutions of problems A, B, C, D.

Let $k=1, k_2=1$. Then from (3.10) we obtain

$$k_{\text{I}(A)} = K_{\text{I}}$$

Let $k=1, k_2=1, \sigma(x)=\sigma\equiv\text{const}$. Then from (3.19), (3.24) we obtain

$$k_{\text{I}(B)} = K_{\text{I}(B)} = \sigma\sqrt{\pi l/2}$$

Similarly, for problems C and D we obtain, for $k=1, k_2=1$,

$$k_{\text{III}(C)} = K_{\text{III}}$$

from (3.26) for problem C, and

$$k_{\text{III}(D)} = K_{\text{III}(D)} = \tau\sqrt{\pi l/2}$$

for problem D from (3.29) and (3.30) for $\tau(x)=\tau\equiv\text{const}$.

We note that in /11, 12/ the solution of problem B was reduced, using the integral Mellin and Fourier transforms, to singular integral first-order equations, with a Cauchy kernel. Numerical methods of solving the analogous problems using integral equations are developed in /13—15/.

REFERENCES

1. KHRAPKOV A.A., First fundamental problem for a piecewise homogeneous plane with a cut perpendicular to the straight-line boundary. PMM Vol.32, No.4, 1968.
2. KULIEV V.D., RABOTNOV YU.N. and CHEREPANOV G.P., On arresting a crack at the interface separating two different media. Izv. Akad. Nauk SSSR, MTT, No.4, 1978.
3. KULIEV V.D., A crack with a finite branch in a piecewise-homogeneous elastic medium. Dokl. Akad. Nauk SSSR, Vol.246, No.6, 1979.
4. KULIEV V.D., Skewing in a longitudinal shear crack. Dokl. Akad. Nauk SSSR, Vol.249, No.2, 1979.
5. KULIEV V.D., Certain problems on branching shear a crack in an elastic piecewise homogeneous medium. Dokl. Akad. Nauk AzSSR, Vol.35, No.3, 1972.
6. LEVERENZ R.K., A finite element stress analysis of crack in a Bi-material plate. Internat. J. Fract. Mech, Vol.8, No.3, 1972.
7. ZAK A.L., and WILLIAMS M.L., Crack point stress singularities at a Bi-material interface. Trans. ASME. Ser. E. J. Appl. Mech., Vol.30, No.1, 1963.
8. UFLYAND YA.S., Integral Transforms in Problems of the Theory of Elasticity. Moscow-Leningrad, Izd-vo Akad. Nauk SSSR, 1963.
9. GAKHOV F.D., Boundary Value Problems. Moscow, Fizmatgiz, 1963.
10. NOBLE B., Methods Based on the Wiener-Hopf Technique for Solving Partial Differential Equations. N.Y. Pergamon Press, 1958.
11. ASHBAUGH N.E., On the opening of a finite crack normal to an interface. Trans. ASME. Ser. E. J. Appl. Mech., Vol.40, No.2, 1973.
12. GOREE J.G. and VENEZIA W.A., Bonded elastic half-planes with an interface crack and a perpendicular intersecting crack that extends into the adjacent material. I, II. Internat. J. Engng Sci., Vol.15, No.1, 1977.
13. ERDOGAN F., GUPTA G.D. and COOK T.S., Numerical solution of singular integral equations. In: Mechanics of fracture. V.1. Methods of analysis and solutions of crack problems. Leyden: Noordhoff Int. Publ. Col., 1973.
14. PANASYUK V.V., SAVRUK M.P. and DATSYSHIN A.P., Distribution of Stresses Around Cracks in Plates and Shells. Kiev, Nauk, Dumka, 1976.
15. PARTON V.Z. and PERLIN P.I., Integral Equations of the Theory of Elasticity. Moscow, Nauka, 1977.

Translated by L.K.